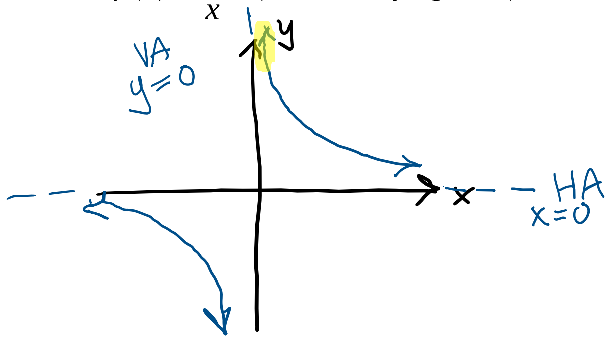
Introduction to Improper Integrals

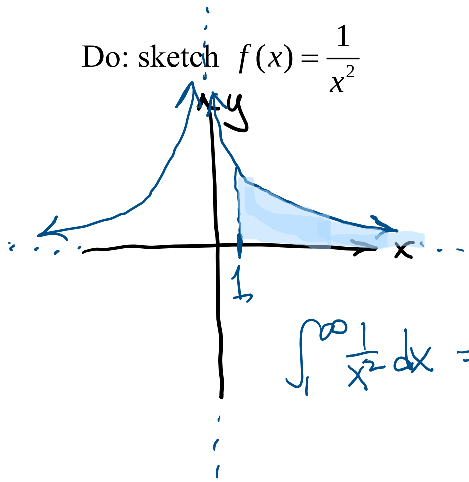
Integral is considered improper if:

- one (or both) bounds are +/− infinity
- integrand is discontinuous within the interval (a, b)

$\int_a^b f(x)\,dx$

Do: sketch $f(x)=\dfrac{1}{x}$ (include asymptotes) then Do: sketch $f(x)=\dfrac{1}{x^2}$





$\int_1^\infty \frac{1}{x^2}\,dx = 1$ ← FINITE

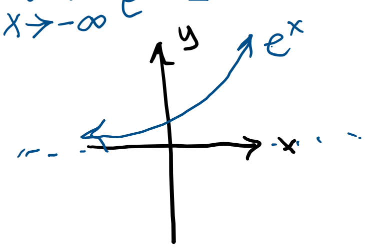
Recall limits involving infinity:

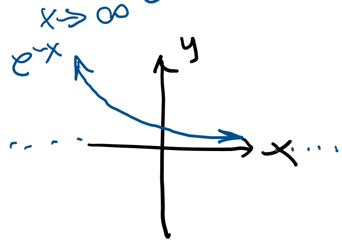
$\lim\limits_{x\to+\infty}\frac{1}{x} = 0$ $\lim\limits_{x\to-\infty}\frac{1}{x} = 0$ $\lim\limits_{x\to 0^+}\frac{1}{x^2} = \infty$ OR DNE

Do: determine the following limits

$\lim\limits_{x\to-\infty} e^{x} = 0$ $\lim\limits_{x\to\infty} e^{-x} = 0$ $\lim\limits_{x\to-\infty} e^{-x} = \infty$ OR DNE

Idea: sketch curve





Recall: indeterminate forms and L'Hospital's Rule

$\frac{\infty}{\infty}$, $\frac{0}{0}$ use L'H

ex. $\lim\limits_{x\to 0} \frac{\sin x}{\tan x} = \frac{\sin 0}{\tan 0} = \frac{0}{0}$

$\xrightarrow{L'H} \lim\limits_{x\to 0} \frac{\cos x}{\sec^2 x} = \frac{1}{1} = \boxed{1}$

$\cos 0 = 1 = \frac{1}{1}$

recip $\updownarrow$

$\sec 0 = 1$

ex. $\lim\limits_{x\to\infty} \frac{e^x}{2x}$ $\frac{\infty}{\infty}$ $\xrightarrow{L'H} \lim\limits_{x\to\infty} \frac{e^x}{2} = \frac{\infty}{2} = \boxed{\infty \text{ or DNE}}$

$0\cdot\infty$ ex. $\lim\limits_{x\to 0^+} x \cdot \ln x$

$0(-\infty)$

write as a fraction

$x = \frac{1}{\frac{1}{x}}$

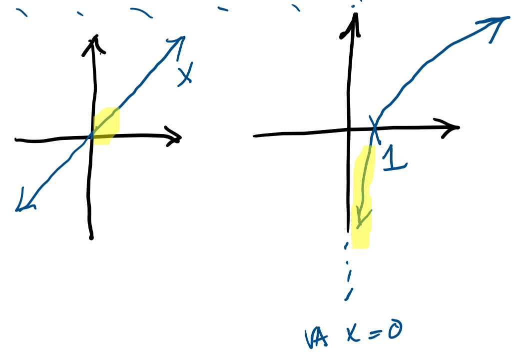
$= \lim\limits_{x\to 0^+} \frac{1}{\frac{1}{x}} \cdot \ln x$

$= \lim\limits_{x\to 0^+} \frac{\ln x}{\frac{1}{x}}$ $\frac{-\infty}{\infty}$ $\xrightarrow{L'H} \lim\limits_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}}$

Do: $\left(\frac{1}{x}\right)' = -\frac{1}{x^2}$

$= -\lim\limits_{x\to 0^+} \frac{1}{x} \cdot \frac{x^2}{1}$

$= -\lim\limits_{x\to 0^+} x = \boxed{0}$



0^∞, ∞^0, 1^∞ $\leftarrow$ use logs to simplify/evaluate

Explore areas under the curve:

recall: $\int_a^b x^n\,dx = \frac{x^{n+1}}{n+1}\Big|_a^b$

ex. $\int_1^2 \frac{1}{x^2}\,dx = \int_1^2 x^{-2}\,dx$

$= \frac{x^{-1}}{-1}\Big|_1^2$

$= -\frac{1}{x}\Big|_1^2 = -\left(\frac{1}{2}-1\right) = -\frac{1}{2}+1$

ex. $\int_1^3 \frac{1}{x^2}\,dx = \int_1^3 x^{-2}\,dx = -\frac{1}{x}\Big|_1^3 = -\left(\frac{1}{3}-1\right) = -\frac{1}{3}+1$

Do: $\int_1^{10} \frac{1}{x^2}\,dx = \int_1^{10} x^{-2}\,dx = -\frac{1}{x}\Big|_1^{10} = -\left(\frac{1}{10}-1\right) = -\frac{1}{10}+1$

Do: $\int_1^{100} \frac{1}{x^2}\,dx = \int_1^{100} x^{-2}\,dx$ $= -\frac{1}{100}+1$

result of integration

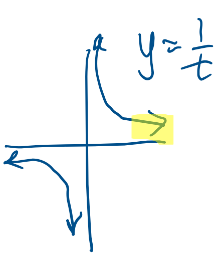
Question: What happens to the area under the curve as the upper bound gets infinitely large? expect area to be 1

introduce limit

ex. $\int_1^{\infty} \frac{1}{x^2}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^2}\,dx$

$= \lim_{t\to\infty} \int_1^t x^{-2}\,dx$

$= \lim_{t\to\infty} \frac{x^{-1}}{-1}\Big|_1^t = -\lim_{t\to\infty} \frac{1}{x}\Big|_1^t$



$= -\lim_{t\to\infty}\left(\frac{1}{t}-1\right)$

$= -\left(\boxed{\cancel{\lim_{t\to\infty}\frac{1}{t}}}^{\,0} - 1\right)$

$= -(-1) = \boxed{1}$

know $\lim_{t\to a} c = c$ where a is a constant c

Definition:

if $\int_a^t f(x)dx$ exists for $t \geq a$ then $\int_a^\infty f(x)dx = \lim_{t\to\infty} f(x)\,dx$

or if $\int_t^b f(x)dx$ exists for $t \leq b$ then $\int_{-\infty}^b f(x)dx = \lim_{t\to-\infty} f(x)\,dx$

this type of integral is convergent if corresponding limit exists

or it is divergent if corresponding limit does not exist (DNE).

assuming both scenarios are convergent then:

$$\int_{-\infty}^{\infty} f(x)\,dx = \underbrace{\int_{-\infty}^{a} f(x)dx}_{\text{converges}} + \underbrace{\int_a^\infty f(x)dx}_{\text{converges}}$$

where a is a constant

ex. consider $\int_1^\infty \frac{1}{x}dx = \lim_{t\to\infty} \int_1^t \frac{1}{x}dx$

$= \lim_{t\to\infty} \ln|x| \Big|_1^t$

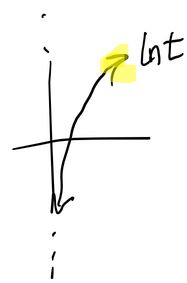
$= \lim_{t\to\infty} (\ln|t| - \cancel{\ln 1})$ → 0

$= \lim_{t\to\infty} \ln|t|$

$= \infty$ OR DNE

$(\ln x)' = \frac{1}{x}$

$\therefore \int_a^b \frac{1}{x}dx = \ln|x| \Big|_a^b$



conclusion: $\boxed{\int_1^\infty \frac{1}{x}dx \text{ diverges}}$

ex. $\int_2^\infty e^{-x}dx = \lim_{t\to\infty} \int_2^t e^{-x}\,dx$

$= -\lim_{t\to\infty} e^{-x} \Big|_2^t$

$= -\lim_{t\to\infty} (e^{-t} - e^{-2})$

$= -\left(\cancel{\lim_{t\to\infty} \frac{1}{e^t}} - \frac{1}{e^2}\right)$ → 0

$= -\left(-\frac{1}{e^2}\right) = \frac{1}{e^2}$

$\lim_{t\to\infty} \frac{1}{\square} = \frac{1}{\text{very large \#}\ (+)} \to$ very small #

$= 0$

$\boxed{\text{Conclusion: } \int_2^\infty e^{-x}dx \text{ converges to } \frac{1}{e^2}}$

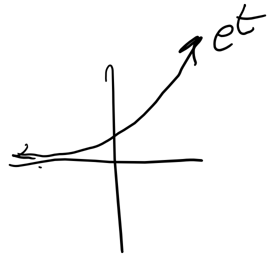
ex. $\int_2^{\infty} e^x dx = \lim_{t\to\infty} \int_2^t e^x\,dx$

$= \lim_{t\to\infty} e^x \Big|_2^t$

$= \lim_{t\to\infty} \left(e^t - e^2\right)$

$= \cancel{\lim_{t\to\infty} e^t}^{\to\infty} - e^2$

$= \infty$



$$\boxed{\text{Conclusion: } \int_2^{\infty} e^x\,dx \text{ diverges}}$$

$\int u\,dv = uv - \int v\,du$

recall: $\overset{u}{\longleftarrow}$ LIATE $\underset{dv}{\longrightarrow}$

ex. $\int_{-\infty}^{0} xe^x dx = \lim_{t\to-\infty} \int_t^0 xe^x\,dx$ use IBP

$u = x \quad dv = e^x dx$

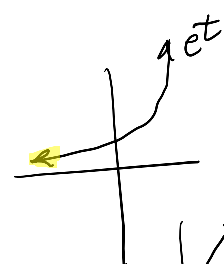
$du = dx \quad v = e^x$

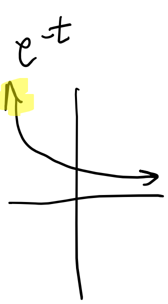
$= \lim_{t\to-\infty} \left(xe^x\Big|_t^0 - \int_t^0 e^x\,dx \right)$

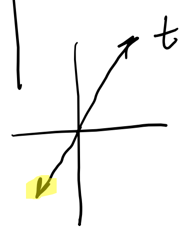
$= \lim_{t\to-\infty} \left(0 - te^t - e^x \Big|_t^0 \right)$

$= \lim_{t\to-\infty} \left(-te^t - (e^0 - e^t) \right)$

$= -\lim_{t\to-\infty} te^t - 1 - \cancel{\lim_{t\to-\infty} e^t}^{\to 0}$







$= -\lim_{t\to-\infty} te^t - 1$

$-\infty(0)$ indeterminate form

$= -\lim_{t\to-\infty} \dfrac{t}{e^{-t}} - 1 \quad \dfrac{-\infty}{\infty}$

$\downarrow$ L'H

$= -\lim_{t\to-\infty} \dfrac{1}{-e^{-t}} - 1$

$= \cancel{\lim_{t\to-\infty} \dfrac{1}{e^{-t}}}^{\to 0} \quad \boxed{-1}$

$$\boxed{\text{conclusion: } \int_{-\infty}^{0} xe^x\,dx \text{ converges to } -1}$$